Representation Theory of Finite Groups
Back-paper exam
June 2024

Time: 3 hours

Max score: 100

Notations: G denotes a finite group throughout, and all representations are over the field of complex numbers.

Answer all questions.

(1) (a) Define representation of a group.
(b) Define irreducible representations.
(c) Let $\phi : G \to GL_2(\mathbb{C})$ be a representation of G. Show that ϕ is irreducible if and only if there is no common eigenvector for the matrices ϕ_g with $g \in G$. (2+2+6)

(2) State and prove Maschke's theorem for representations of a finite group. (15)

(3) State and prove Schur's lemma. (15)

(4) (a) Define character χ_ϕ of a representation ϕ of a group G.
(b) Define direct sum $\phi \oplus \psi$ of two representations ϕ and ψ of G.
(c) Show that if $\rho = \phi \oplus \psi$ then $\chi_\rho = \chi_\phi + \chi_\psi$. (2+3+5)

(5) (a) Define regular representation of a group G.
(b) Find the character of the regular representation.
(c) Show that if $\phi^1, \ldots, \phi^s$ denote a complete set of representatives of the equivalence classes of irreducible representations of G, then the multiplicity of ϕ^i in the decomposition of the regular representation is $\deg(\phi^i)$, for all $1 \leq i \leq s$. (4+8+8)

(6) Write down the character table for S_4, with proper justifications. (10)

(7) Let G be a non-abelian group of order 21.
(a) Determine the degrees of the irreducible representations of G
(b) How many irreducible representations G has of each degree (up to equivalence)?
(b) Determine the number of conjugacy classes of G. (4+4+2)

(8) (a) Define Specht representation S^λ of the symmetric group S_n corresponding to the partition λ of n.
(b) Prove that the Specht representation corresponding to the partition $\lambda = (n-1, 1)$ of n is the standard representation of S_n.
(6+4)

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